UECM1303 Tutorial 4: Information Structures

May 2021

Set Relations, Representations & Properties

1. Let $A = \{a \in \mathbb{R} | -2 \leq a \leq 3\}$ and $B = \{b \in \mathbb{R} | 1 \leq b \leq 5\}$. Sketch the given set in the Cartesian plane $\mathbb{R}^2$ for (i) $A \times B$; (b) $B \times A$.

2. Define a relation R on $\mathbb{R}$ as follows:

   xRy if and only if x, y satisfy the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

   (a) Which of the following ordered pairs belong to R?
      i. (2, 0) .... ☐
      ii. (0, 2) .... ☐
      iii. (0, 3) .... ☐
      iv. (0, 0) .... ☐
      v. $(1, \frac{3\sqrt{3}}{2})$ .... ☐

   (b) Find $R(\{1, 7\})$ and $R(\{3, 4, 5\})$.

3. Find the domain, range, matrix representation of the relation R.

   (a) $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, $R = \{(a, 1), (a, 2), (b, 1), (c, 2), (d, 1)\}$.
   (b) $A = \{1, 2, 3, 4\}$, $B = \{1, 4, 6, 9\}$; aRb if and only if $b = a^2$.
   (c) $A = \{1, 2, 3, 4, 8\}$, $B = \{1, 4, 6, 9\}$; aRb if and only if a divides b.
   (d) $A = \{1, 2, 3, 4, 5\} = B$; aRb if and only if $a \leq b$.

4. Suppose R and S are reflexive relations on a set A. Prove or disprove each of the following:

   (a) $R \cup S$ is reflexive.
   (b) $R \cap S$ is reflexive.
   (c) $S \circ R := \{(a, c) : \exists b((a, b) \in R \wedge (b, c) \in S\}$ is reflexive.

5. Give an example of a relation on a set that is

   (a) symmetric and anti-symmetric.
   (b) neither symmetric nor anti-symmetric.

Closure of Binary Relations

6. Let $A = \{a, b, c, d, e\}$ and R and S be the relations on A described by

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad M_S = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

   Use Warshall's algorithm to compute the transitive closure of the relation $R \cup S$.

7. Let R be a relation on the set $A = \{a_1, a_2, a_3, a_4, a_5\}$ with a matrix representation:

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

(a) Write down the listing tuples (or Roster notation) representation of R.

(b) Compute $M_{cl_{trn}(R)}$ as in Warshall's algorithm and then sketch the digraph representation of $cl_{trn}(R)$.

(c) Is R transitive? Explain your answer.

Equivalence Relations

8. In each of the following, determine whether the relation is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive. Hence, determine which of the following relation on the set A is an equivalence relation.

(a) $A = \{1, 2, 3, 4\}$,

i. $R = \{(1, 1), (2, 2), (3, 3)\}$

ii. $R = \emptyset$

iii. $R = A \times A$

iv. $M_R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(b) $A = \mathbb{Z}$; $aRb \Leftrightarrow a \leq b + 1$.

(c) $A = \mathbb{Z}$; $xRy \Leftrightarrow |x - y| \leq 2$.

(d) $A = \mathbb{R}$; $aRb \Leftrightarrow a^2 + b^2 = 4$.

9. If R and S are two relations on $\mathbb{R}$ such that for $x, y \in \mathbb{R}$, xRy iff $x < y$ and xSy iff $x > y$. Find (i) $R \cap S$ (ii) $R \cup S$ (iii) $S^{-1} := \{(y, x) : (x, y) \in S\}$.

10. Let $A = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$ and define a relation R on A as follows:

$$\forall (a, b) \in A, \forall (c, d) \in A, (a, b)R(c, d) \leftrightarrow ab = cd.$$

(a) Verify that R is an equivalence relation on A.

(b) Determine the equivalence class $[(2, 3)]$ by listing all its elements.

11. Let R be the relation on $A = \{2, 4, 6, 8\}$ defined by $xRy \leftrightarrow \gcd(x, y) = 2$.

(a) Write R as a set of ordered pairs.

(b) Determine whether R is an equivalence relation.

12. Given $M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$. Compute A/R.

13. Let $S = \{1, 2, 3, 4\}$ and $A = S \times S$. Define the equivalence relation R on A as follows: $(a, b)R(c, d)$ if and only if $a + b = c + d$. Compute A/R.

14. Define a binary relation R on $\mathbb{R}$ as follows:

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} | y = |x|\}.$$

Determine whether R is reflexive, symmetric and transitive.

15. Let R be an equivalence relation on $\mathbb{Z}$ such that for $x, y \in \mathbb{Z}$, xRy iff $7|x - y$. Which of the following equivalence classes

$$[3], [-7], [12], [0], [-2], [17]$$

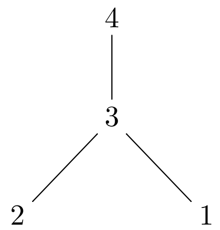
are equal?

Partial Order Relations

16. Determine whether the relation R is a partial order on $\mathbb{Z}$.

(a) aRb if and only if $a = 3b$.
(b) aRb if and only if $a^2|b$.
(c) aRb if and only if $a = b^k$ for some positive integers k.

17. Describe the ordered pairs in the relation $\preceq$ determined by the Hasse diagram on the set $A = \{1, 2, 3, 4\}$.



18. Consider the poset $(A, |)$ with $|$ the divisibility relation. Draw the Hasse diagram of the poset and determine which posets are linearly ordered.

(a) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$
(b) $A = \{3, 6, 12, 72\}$
(c) $A = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 30, 60\}$.

19. Let $\preceq$ be a relation over the set $A = \{1, 2, 3, 4, 5\}$ such that

$$1 \preceq 1,\ 1 \preceq 2,\ 1 \preceq 3,\ 1 \preceq 4,\ 1 \preceq 5,\ 2 \preceq 2,\ 2 \preceq 5,\ 3 \preceq 3,\ 3 \preceq 5,\ 4 \preceq 4,\ 4 \preceq 5,\ 5 \preceq 5.$$

Show that $(A, \preceq)$ is a poset.