UECM1303 Tutorial 2 Solution

May 2021

Predicates & Quantified Statements

1. Let P be a proposition and $Q(x,y)$ be a predicate. Are the following strings well-formed formula?

   (a) $\sim$ .... No. Lack of a formula on the right.

   (b) $(\sim (P)) \wedge (P)$ .... Yes.

   (c) $\forall x((P \to Q(x,y)) \vee (P))$ .... Yes.

   (d) $\sim \forall x Q(x,x^2) \equiv \exists x \sim Q(x,x^2)$ .... No. $\equiv$ is not allowed in a formula.

   (e) $x^2 + y^2 - 3z^2$ .... No. It is just a term

   (f) $x^2 + y^2 - 3z^2 = 0$ .... Yes. It is a predicate, which is a formula

2. Let a be a constant, f_i be functions and P_i be predicates. Determine the bound and free variables of the following formula.

   (a) $(\forall x_2 P_1(x_1, x_2)) \to P_1(x_2, a)$. .... Free: x_1, Second x_2, Bound: First x_2

   (b) $P_1(x_3) \to \sim \forall x_1 \forall x_2 P_2(x_1, x_2)$. .... Free: x_3, Bound: x_1, x_2

   (c) $\forall x_2(P_1(f_1(x_1, x_2), x_1) \to \forall x_1 P_2(x_3, f_2(x_1, x_2)))$ .... Free: x_3, First x_1, Bound: x_2, Second x_1

3. Consider the predicates

$LT(x,y) : x < y$	$EQ(x,y) : x = y$	$EV(x)$: x is even
$I(x)$: x is an integer	$P(x) : x > 0$	$R(x)$: $x \in \mathbb{R}$.

   (a) Write the statement using ONLY these predicates and any needed quantifiers.

      i. Every integer is even. .... $\forall x(I(x) \to EV(x))$

      ii. Some real numbers are negative integers. .... $\exists x(R(x) \wedge I(x) \wedge \sim P(x) \wedge \sim EQ(x,0))$

      iii. If $x < y$, then x is not equal to y. ...... $\forall x \forall y\ LT(x,y) \to \sim EQ(x,y)$

      iv. There is no largest number. .... $\sim \exists x \forall y(LT(y,x) \vee EQ(y,x))$

      v. If $y > x$ and $0 > z$, then $x \cdot z > y \cdot z$. .... $\forall x \forall y \forall z(LT(x,y) \wedge LT(z,0) \to LT(y \cdot z, x \cdot z))$

   (b) Write the statement $\exists x \sim P(x)$ in English without using variables and symbols. ... Some real numbers are not positive.

4. Let $M(s)$ denote "s is a math major", $C(s)$ denote "s is a computer science student" and $E(s)$ denote "s is an engineering student". Rewrite the following statements by using quantifiers, variables and predicates $M(s)$, $C(s)$ and $E(s)$.

(a) There is an engineering student who is a math major. .. $\exists x(E(x) \wedge M(x))$

(b) Every computer science student is an engineering student. $\forall x(C(x) \rightarrow E(x))$

(c) No computer science students are engineering students. . $\sim \exists x(C(x) \wedge E(x))$

(d) Some computer science students are engineering students and some are not.
.... $(\exists x(C(x) \wedge E(x))) \wedge (\exists x(C(x) \wedge \sim E(x)))$

5. Let $H(x)$ denote the predicate "x is a human" and $T(x, y)$ denote the predicate "x trusts y". Rewrite the following formula into English sentence without quantifiers and variables.

(a) $\forall x \exists y(H(x) \rightarrow (H(y) \wedge T(x, y)))$ .... Everybody trusts somebody.

(b) $\exists x \forall y(H(x) \wedge (H(y) \rightarrow \sim T(x, y)))$. .... Some people trusts nobody.

(c) $\forall x \forall y(H(x) \rightarrow (H(y) \rightarrow \sim T(x, y))$. .... Everybody does not trust everybody.

6. Consider the following statement:

$$\exists x(x \in \mathbb{R} \wedge x^2 = 2).$$

Which of the following are equivalent ways of expressing this statement?

(a) The square of each real number is 2. .... No

(b) Some real numbers have square 2. .... Yes

(c) If x is a real number, then $x^2 = 2$. .... No

7. (Difficult, requires logic programming) Express the following summation using predicate:

$$S(n) = 1^2 + 2^2 + \cdots + n^2.$$

Ans.
- thesum(1, 1).
- $n > 1 \wedge$ thesum($n - 1$, S_1) $\wedge S = S_1 + n^2 \rightarrow$ thesum(n, S).

Logical Equivalence

8. Determine whether each of the following statements is true or false over the model of integer sets.

   (a) $\forall x \exists y \exists z (x = 7y + 5z)$ .... Let $y = -2x$ and $z = 3x$ .... T

   (b) $\forall x \exists y \exists z (x = 31y + 41z)$ .... Let $y = 4x$ and $z = -3x$ .... T

   (c) $\forall x \exists y \exists z (x = 4y + 6z)$ .... Let $x = 5$, $\forall y \forall z (5 \neq 4y + 6z = 2(2y + 3z))$ . . F

9. Determine the truth value of the following statements assuming we are interpreting these formulae over the real number domain.

   (a) $\forall x (x > \frac{1}{x})$. .... False. $x = 0.2$

   (b) $\exists x (x \in \mathbb{Z} \to \frac{x-3}{x} \notin \mathbb{Z})$. .... True. $x = 0.5$

   (c) $\exists m \exists n (m \in \mathbb{Z} \wedge n \in \mathbb{Z} \wedge m > 0 \wedge n > 0 \wedge mn \geq m + n)$. .... True. $m = n = 3$

   (d) $\forall x \forall y (\sqrt{x+y} = \sqrt{x} + \sqrt{y})$. .... False. $x = 2$, $y = 3$.

10. Let $E(n)$ be the predicate "n is even" and consider the following statement:

$$\forall n (n \in \mathbb{Z} \to (E(n^2) \to E(n))).$$

Which of the following are equivalent ways of expressing this statement?

   (a) All integers have even squares and are even. .... No

   (b) Given any integer whose square is even, that integer is itself even. .... Yes

   (c) For all integers, there are some whose square is even. .... No

   (d) Any integer with even square is even. .... Yes

   (e) All even integers have even squares. .... No

11. Give a negation for each statement below:

   (a) For all integers x, if x is odd, then $x^2 - 1$ is even.

   (b) There exists an integer x with $x \geq 2$ such that $x^2 - 4x + 7$ is prime.

   (c) For all real numbers x and y, if $x = y$, then $x^2 = y^2$.

   (d) There is no easy question in the exam.

   (e) If the square of real number x is greater than or equal to 1 then $x > 0$.

Ans. (a) There exists an integer x such that x is odd and $x^2 - 1$ is not even.

(b) For each integer x with $x \geq 2$, $x^2 - 4x + 7$ is not prime.

(c) There are real numbers x and y such that $x = y$ but $x^2 \neq y^2$.

(d) There are some easy questions in the exam.

(e) There is a real number x such that $x^2 \geq 1$ but $x \leq 0$.

12. Show that the statements $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x (P(x) \wedge Q(x))$ are not logically equivalent in general.

Ans. Consider the model M with the universe of discourse $\mathbb{Z}$ and $P(x)$ be "$x > 0$" and $Q(x)$ be "$x < 0$".

$\exists x P(x) \wedge \exists x Q(x)$ is true whereas $\exists x P(x) \wedge Q(x)$ is false.

13. Determine whether the statements $\forall x P(x) \wedge \exists x Q(x)$ and $\forall x \exists y (P(x) \wedge Q(y))$ are logically equivalent.

Ans. Under any interpretation/model with domain D, if $\forall x P(x) \wedge \exists x Q(x)$ is true, $P(x)$ is true for all x and $Q(x)$ is true from some $x = y$. So for all x, there is a y such that $P(x)$ is true and $Q(y)$ is true. Hence, $\forall x \exists y (P(x) \wedge Q(y))$ is true.

On the other hand, if $\forall x \exists y (P(x) \wedge Q(y))$ is true, then for all x, there is a y such that $P(x)$ is true and $Q(y)$ is true, so $\forall x, P(x)$ and $\exists y Q(y)$ are true. Hence, $(\forall x P(x)) \wedge (\exists y Q(y)) \equiv T$. Changing the symbol y to x leads a logical equivalence between the two statements.

14. Determine whether $\exists x (P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ have the same truth value. Explain.

Ans. Under any interpretation/model with domain D, $\exists x (P(x) \vee Q(x))$ is true means that $P(s) \vee Q(s) \equiv T$ for some $s \in D$. By definition, either $P(s) \equiv T$ or $Q(s) \equiv T$. This means $\exists x P(x) \equiv T$ or $\exists x Q(x) \equiv T$. Hence, $\exists x P(x) \vee \exists x Q(x)$ is true.

Conversely, if $\exists x P(x) \vee \exists x Q(x) \equiv T$, then $\exists x P(x) \equiv T$ or $\exists x Q(x) \equiv T$. This means, there is an s_1 such that $P(s_1) \equiv T$, which implies $P(s_1) \vee Q(s_1) \equiv T \vee Q(s_1) \equiv T$. Similarly, there is an s_2 such that $Q(s_2) \equiv T$, which implies $Q(s_2) \vee P(s_2) \equiv T$. In either case, $\exists x (P(x) \vee Q(x)) \equiv T$.

15. Determine the truth value of each of these statements if they are modelled over the set of integers.

(a) $\forall n \exists m (n^2 < m)$ .... True. Take $m = n^2 + 1$

(b) $\exists n \forall m (nm = m)$ .... True. Take $n = 1$

(c) $\exists n \exists m (n^2 + m^2 = 6)$ .... False. Try all integers ≤ 2

16. Let $P(x, y)$ be a predicate and the domain of discourse be a nonempty set. Given that $\forall x \exists y P(x, y)$ is true, which of the following are not necessarily true?

(a) $\exists x \exists y P(x, y)$ .... ×

(b) $\forall x \forall y P(x, y)$ .... ✓

(c) $\exists x \forall y P(x, y)$ .... ✓

17. What are the truth values of $\exists y \forall x (y \geq x)$ and $\forall x \exists y (y \geq x)$ if they are interpreted over the model with the domain of nonnegative integers?

Ans. $\exists y \forall x (y \geq x)$ is false because for every non-negative integer y, there is an integer x such that $y < x$. For example, take $x = y + 1$.

$\forall x \exists y (y \geq x)$ is true because for every non-negative integer x, there is an integer y such that $y < x$. For example, take $y = x + 1$.

18. Let odd(x) be the predicate "x is an odd positive integer." Determine the truth value of each of the following statements for the model M with domain of positive integers. If the statement is false, provide an explanation or suggest a counterexample.

(a) $\forall x \forall y (\text{odd}(x + y))$. .... False. When $x = 2$, $y = 4$, $x + y = 2 + 4 = 6$ is even.

(b) $\exists x \forall y (\text{odd}(x + y))$. .... False. If x is even(odd), add even(odd) y.

(c) $\forall x \exists y (\text{odd}(x + y))$. .... True. If x is even(odd), add odd(even) y.

(d) $\exists x \exists y(xy + 1 = 0)$. .... False. No positive numbers add 1 can be 0.

19. Consider the following models

$$M_1 = (\mathbb{N}, \{=\}, \{+^{\mathbb{N}}, \cdot^{\mathbb{N}}\}, 0^{\mathbb{N}}) \text{ where}$$

$$\begin{aligned}
&(=^{\mathbb{N}}) = \{(n,n) | n \in \mathbb{N}\},\\
&+^{\mathbb{N}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (m,n) \mapsto m+n,\\
&\cdot^{\mathbb{N}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (m,n) \mapsto m \cdot n.
\end{aligned}$$

$$M_2 = (\mathbb{Q}, \{=\}, \{+^{\mathbb{Q}}, \cdot^{\mathbb{Q}}\}, 0^{\mathbb{Q}}) \text{ where}$$

$$\begin{aligned}
&(=^{\mathbb{Q}}) = \{(n,n) | n \in \mathbb{Q}\},\\
&+^{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, \quad (m,n) \mapsto m+n,\\
&\cdot^{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, \quad (m,n) \mapsto m \cdot n.
\end{aligned}$$

Determine the truth value of

(a) $\forall x_1 \forall x_2 \exists x_3 (x_1 + x_2 = x_3)$ and

Ans. Let $\sigma_1 = \{(x_1, a), (x_2, b), (x_3, a+b)\}$ where a, b are arbitrary values from $\mathbb{N}$. Then

$$(\forall x_1 \forall x_2 \exists x_3 (x_1 + x_2 = x_3))^{M_1}(\sigma_1) \equiv a + b = a + b \equiv T.$$

This is similar for M_2, just change $\mathbb{N}$ to $\mathbb{Q}$.

(b) $\forall x_1 \forall x_2 ((\exists x_3 (x_1 \cdot x_3 = x_2) \wedge \exists x_4 (x_2 \cdot x_4 = x_1) \to x_1 = x_2)$.

Ans. Let $\sigma_2 = \{(x_1, a), (x_2, b)\}$ where a, b are two arbitrary different values from $\mathbb{N}$. Then

$$\begin{aligned}
&(\forall x_1 \forall x_2 ((\exists x_3 (x_1 \cdot x_3 = x_2) \wedge \exists x_4 (x_2 \cdot x_4 = x_1) \to x_1 = x_2))^{M_1}(\sigma_2)\\
\equiv&(\exists x_3 (a \cdot x_3 = b)) \wedge (\exists x_4 (b \cdot x_4 = a)) \to a = b\\
\equiv&\exists x_3 \exists x_4 ((a \cdot x_3 = b) \wedge (b \cdot x_4 = a)) \to a = b\\
\equiv&\exists x_3 \exists x_4 ((a \cdot x_3 = b) \wedge (b \cdot x_4 = a) \wedge b \cdot x_4 \cdot x_3 = b) \to a = b\\
\equiv&\exists x_3 \exists x_4 ((a \cdot x_3 = b) \wedge (b \cdot x_4 = a) \wedge \underline{b \cdot x_4 \cdot x_3 = b}) \to a = b\\
\equiv&\exists x_3 \exists x_4 ((a \cdot x_3 = b) \wedge (b \cdot x_4 = a) \wedge \underline{x_4 \cdot x_3 = 1}) \to (a = b)\\
\equiv&F \to F \equiv T
\end{aligned}$$

Note that for natural numbers, $x_4 \cdot x_3 = 1$ implies $x_4 = x_3 = 1$. Hence, no such x_3 and x_4 exist above.
Assuming a, b are two arbitrary different non-zero values from $\mathbb{Q}$ and let $\sigma_3 = \{(x_1, a), (x_2, b), (x_3, b/a), (x_4, a/b)\}$. Then

$$\begin{aligned}
&(\forall x_1 \forall x_2 ((\exists x_3 (x_1 \cdot x_3 = x_2) \wedge \exists x_4 (x_2 \cdot x_4 = x_1) \to x_1 = x_2))^{M_2}(\sigma_3)\\
\equiv&(\exists x_3 (a \cdot x_3 = b)) \wedge (\exists x_4 (b \cdot x_4 = a)) \to a = b\\
\equiv&T \wedge T \to F \equiv F.
\end{aligned}$$

20. Derive the following rule using laws of equivalence:

$$\sim (\forall x (x \in D \to (\forall y (y \in E \to P(x,y))) \equiv \exists x \exists y (x \in D \wedge (y \in E \wedge \sim P(x,y))$$

$Ans.$ $\sim (\forall x(x \in D \to (\forall y(y \in E \to P(x,y)))$

$\equiv \exists x \sim (x \in D \to (\forall y(y \in E \to P(x,y)))$ [GDM]

$\equiv \exists x \sim (\sim (x \in D) \vee (\forall y(y \in E \to P(x,y)))$ [Implication Law]

$\equiv \exists x((x \in D) \wedge \sim (\forall y(y \in E \to P(x,y))))$ [De Morgan & Double Negation]

$\equiv \exists x((x \in D) \wedge (\exists y \sim ((\sim (y \in E) \vee P(x,y)))))$ [GDM & Implication Law]

$\equiv \exists x((x \in D) \wedge (\exists y((y \in E) \wedge \sim P(x,y))))$ [De Morgan & Double Negation]

$\equiv \exists x \exists y((x \in D) \wedge ((y \in E) \wedge \sim P(x,y)))$ [Quantified conjuction]

21. Show that $\forall x[(C(x) \wedge \exists y(T(y) \wedge L(x,y))) \to \exists y(D(y) \wedge B(x,y))] \equiv \forall x \forall y \exists z[(C(x) \wedge T(y) \wedge L(x,y)) \to (D(z) \wedge B(x,z))]$

$Ans.$

$\forall x[(C(x) \wedge \exists y(T(y) \wedge L(x,y))) \to \exists y(D(y) \wedge B(x,y))]$

$\equiv \forall x[\underline{(C(x) \wedge \exists y(T(y) \wedge L(x,y)))} \to \exists z(D(z) \wedge B(x,z))]$ [change variable name]

$\equiv \forall x\ [\underline{\exists y(C(x) \wedge (T(y) \wedge L(x,y)))} \to \exists z(D(z) \wedge B(x,z))]$ [Quantified conjuction]

$\equiv \forall x\ [\sim \underline{\exists y(C(x) \wedge (T(y) \wedge L(x,y)))} \vee \exists z(D(z) \wedge B(x,z))]$ [implication law]

$\equiv \forall x\ [\underline{\forall y \sim (C(x) \wedge (T(y) \wedge L(x,y)))} \vee \exists z(D(z) \wedge B(x,z))]$ [GDM]

$\equiv \forall x\ [\forall y[\underline{\sim (C(x) \wedge (T(y) \wedge L(x,y))) \vee \exists z(D(z) \wedge B(x,z))}]]$ [Quantified disjunction]

$\equiv \forall x\ [\forall y[\exists z[\underline{\sim (C(x) \wedge (T(y) \wedge L(x,y))) \vee (D(z) \wedge B(x,z))}]]]$ [Quantified disjunction]

$\equiv \forall x \forall y \exists z[(C(x) \wedge T(y) \wedge L(x,y)) \to (D(z) \wedge B(x,z))]$ [implication law]

22. Write a negation for the following statement:

For all real numbers $y > 0$, there exists a real number $z > 0$ such that if $a - z < x < a + z$ then $L - y < f(x) < L + y$.

$Ans.$ Formally

$\forall y(y > 0 \to (\exists z((z > 0) \wedge ((a-z < x) \wedge (x < a+z) \to (L-y < f(x)) \wedge (f(x) < L+y)).$

It's negation is

$\exists y(y > 0 \wedge (\forall z((z > 0) \to ((a-z < x) \wedge (x < a+z) \to ((L-y \geq f(x)) \vee (f(x) \geq L+y))).$

Informally, the negation is "There is a real numbers $y > 0$ such that for all real numbers $z > 0$, $a - z < x < a + z$ and either $L - y \geq f(x)$ or $f(x) \geq L + y$."

Logical Implication

23. Show that $\sim (\forall x(P(x) \to Q(x))) \Rightarrow \exists x(P(x) \wedge \sim Q(x))$.

    Note that the two quantified statements are actually logically equivalent. Are you able to show the reverse?

    Ans.

1	$\sim (\forall x(P(x) \to Q(x)))$	premise
2	$\exists x \sim (P(x) \to Q(x)))$	1, Generalised De Morgan
3	$\sim (P(s) \to Q(s)))$	2, Existential Instantiation
4	$\sim (\sim P(s) \vee Q(s)))$	3, implication law
5	$P(s) \vee \sim Q(s)$	4, De Morgan & Double Negation
6	$\exists x(P(x) \vee \sim Q(x))$	5, Existential generalisation

24. For the following arguments, state which are valid and which are invalid. Justify your answers.

    (a) All healthy people eat an apple a day. John is not a healthy person. Therefore John does not eat an apple a day.

    Ans. Let $H(x)$ be "x is a healthy person" and $A(x)$ be "x eats an apple a day." Formally, we have

    $$\forall x(H(x) \to A(x)),\ \sim H(\text{John}) / \therefore \sim A(\text{John})$$

    Consider a model M with $D = \{j, a\}$, where j means John and a means Adam, $H^M = \{a\}$ and $A^M = \{a, j\}$. Hence, the interpretation is

    $$\begin{aligned}
    (\forall x(H(x) \to A(x)))^M &\equiv (H^M(a) \to A^M(a)) \wedge (H^M(j) \to A^M(j)) \\
    &\equiv (T \to T) \wedge (F \to T) \equiv T \\
    (\sim H(\text{John}))^M &\equiv \sim H^M(j) \equiv \sim F \equiv T \\
    \hline
    (\sim A(\text{John}))^M &\equiv \sim A^M(j) \equiv \sim T \equiv F.
    \end{aligned}$$

    This indicates that we have $T \to F$ in an argument, hence, the argument is invalid.

    (b) Every student who studies discrete mathematics is good at logic. John studies discrete mathematics. Therefore John is good at logic.

    Ans. Let $D(x)$ be "student x studies discrete mathematics" and $L(x)$ be "x is good at logic". Formally, we have

    $$\forall x(D(x) \to L(x)),\ D(\text{John}) / \therefore L(\text{John})$$

1	$\forall x(D(x) \to L(x))$	premise
2	$D(\text{John})$	premise
3	$D(\text{John}) \to L(\text{John})$	1, universal instantiation
4	$L(\text{John}))$	2,3, MP

    Hence, the argument is valid.

    (c) No heavy object is cheap. XYZ is not a heavy object. Therefore XYZ is cheap.

Ans. Let $H(x)$ be "x is a heavy object" and $C(x)$ be "x is cheap." Formally, we have

$$\forall x(H(x) \to \sim C(x)),\ \sim H(\text{XYZ}) / \therefore C(\text{XYZ})$$

Consider a model M with $D = \{h, d\}$, where h means a house and d means a diamond, $H^M = \{h\}$ and $C^M = \{\}$ and $(\text{XYZ})^M = d$. Hence, the interpretation is

$$\begin{array}{ll} (\forall x(H(x) \to \sim C(x)))^M & \equiv (H^M(h) \to \sim C^M(h)) \wedge (H^M(d) \to \sim C^M(d)) \\ & \equiv (T \to (\sim F)) \wedge (F \to (\sim F)) \equiv T \\ (\sim H(\text{XYZ}))^M & \equiv \sim H^M(d) \equiv \sim F \equiv T \\ \hline (C(\text{XYZ}))^M & \equiv C^M(d) \equiv F. \end{array}$$

This indicates that we have $T \to F$ in an argument, hence, the argument is invalid.

25. Show that the following argument is valid using the laws of logical equivalence and logical implication.

$$\begin{array}{ll} & \exists x(F(x) \wedge S(x)) \to (\forall y(M(y) \to W(y))) \\ & \exists y(M(y) \wedge \sim W(y)) \\ \hline \therefore & \forall x(F(x) \to \sim S(x)) \end{array}$$

Ans.

1	$\exists x(F(x) \wedge S(x)) \to (\forall y(M(y) \to W(y)))$	premise
2	$\exists y(M(y) \wedge \sim W(y))$	premise
3	$\sim \forall y \sim (M(y) \wedge \sim W(y))$	2, negation^2 & GDM
4	$\sim \forall y(\sim M(y) \vee W(y))$	4, DM & double neg.
5	$\sim \forall y(M(y) \to W(y))$	5, DM & implication law
6	$\sim (\exists x(F(x) \wedge S(x)))$	1,5, MT
7	$\forall x(\sim F(x) \vee \sim S(x))$	6, GDM & DM
8	$\forall x(F(x) \to \sim S(x))$	7, implication law

Rules of Inference

26. What is wrong with the following proof?

1	$\forall x \exists y (x > y)$	premise
2	$\exists y (c > y)$	universal instantiation, c arbitrary
3	$(c > s)$	existential instantiation, s specific
4	$\forall x (x > s)$	universal generalisation
5	$\exists y \forall x (x > y)$	existential generalisation

Ans. Step 4 is wrong since s is dependent on c and the universal generalisation will affect s as well.

27. Use ONLY the rules of inference to show that

$$\forall x (P(x) \to (Q(x) \land R(x))),\ \forall x (P(x) \land S(x)) \ \vdash\ \exists x (R(x) \land S(x))$$

Ans.

1	$\forall x (P(x) \to Q(x) \land R(x))$	premise
2	$\forall x (P(x) \land S(x))$	premise
3	$s,\ R(s) \land S(s)$	assumption
4	$P(s) \land S(s)$	2, $\forall$-elimination
5	$P(s) \to Q(s) \land R(s)$	1, $\forall$-elimination
6	$P(s)$	3, $\land$-elimination1
7	$Q(s) \land R(c)$	4,5, $\to$-elimination
8	$R(s)$	6, $\land$-elimination2
9	$S(s)$	3, $\land$-elimination2
10	$R(s) \land S(s)$	7,8, $\land$-introduction
11	$\exists x (R(x) \land S(x))$	3, 10, $\exists$-introduction

28. Use rules of inference to show that

$$\exists x P(x) \to \forall x (P(x) \lor Q(x) \to R(x)),\ \exists x (P(x) \land Q(x)) \vdash \exists y R(y)$$

Ans.

1	$\exists x P(x) \to \forall x (P(x) \vee Q(x) \to R(x))$	premise
2	$\exists x (P(x) \wedge Q(x))$	premise
3	$s,\ P(s) \wedge Q(s)$	assumption
4	$P(s)$	3, $\wedge$-elimination1
5	$\exists x P(x)$	3, 4, $\exists$-introduction
6	$\forall x (P(x) \vee Q(x) \to R(x))$	1,5, $\to$-elimination
7	$s,\ P(s) \wedge Q(s)$	assumption
8	$P(s) \vee Q(s) \to R(s)$	7, $\forall$-elimination
9	$P(s) \vee Q(s)$	4, $\vee$-introduction
10	$R(s)$	8,9 $\to$-elimination
11	$\exists y R(y)$	7,10 $\exists$-introduction

29. Show that $\forall x (Q(x) \to R(x)) \wedge (\exists x (Q(x) \wedge I(x)) \vdash \exists x (R(x) \wedge I(x))$.

Ans.

1	$\forall x (Q(x) \to R(x)) \wedge (\exists x (Q(x) \wedge I(x)))$	premise
2	$\forall x (Q(x) \to R(x))$	1, $\wedge$-elimination1
3	$\exists x (Q(x) \wedge I(x))$	1, $\wedge$-elimination2
4	$s,\ Q(s) \wedge I(s)$	assumption
5	$Q(s) \to R(s)$	2, $\forall$-elimination
6	$Q(s)$	4, $\wedge$-elimination1
7	$R(s)$	5,6, $\to$-elimination
8	$I(s)$	4, $\wedge$-elimination2
9	$R(s) \wedge I(s)$	7,8, $\wedge$-introduction
10	$\exists x (R(x) \wedge I(x))$	4,9, $\exists$-introduction

30. Prove that the following argument is valid using the rules of inference: "Every undergraduate is either an arts student or a science student. Some undergraduates are top students. James is not a science student, but he is a top student. Therefore if James is an undergraduate, he is an arts student."

Ans. Let $U(x)$ be the predicate "x is an undergraduate", $A(x)$ be the predicate "x is an arts student", $S(x)$ be the predicate "x is a science student" and $T(x)$ be the predicate "x is a top student." Then

1	$\forall x(U(x) \to A(x) \vee S(x))$	premise
2	$\exists x(U(x) \wedge T(x))$	premise
3	$\sim S(\text{James}) \wedge T(\text{James})$	premise
4	$U(\text{James})$	assumption
5	$U(\text{James}) \to A(\text{James}) \vee S(\text{James})$	1, universal instantiation
6	$A(\text{James}) \vee S(\text{James})$	4,5, $\to$-elimination
7	$\sim S(\text{James})$	3, $\wedge$-elimination1
8	$A(James)$	assumption
9	$A(James)$	8
10	$S(James)$	assumption
11	$\perp$	7,10 $\neg$-elimination
12	$A(James)$	11, $\neg$-introduction
13	$A(\text{James})$	6,8–12 $\vee$-elimination
14	$U(\text{James}) \to A(\text{James})$	4–13, $\to$-introduction

31. Write a Prolog program to apply the predicate in Question 7 to perform calculations for $n = 1$, $n = 10$ and $n = 50$.

Ans. thesum(1,1).

thesum(N,S) :- N>1, thesum(N-1, S1), S is S1+n^2.