UECM1304 Test 2 Marking Guide

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Course Code & Course Title: UECM1304 Discrete Mathematics with Applications
Faculty: LKC FES, UTAR Course: AM
Trimester: Jun 2025 Lecturer: Liew How Hui

Instruction: Answer all questions in the space provided. **If you do not write your answer in the space provided, you will get ZERO mark**. An answer without working steps may also receive ZERO mark.

1. CO3: Demonstrate various proof-techniques. C3

   (a) Use direct proof to show that there are integers m and n such that $2674m + 5257n = 7$. [**Note**: You need to use extended Euclidean algorithm to find out m and n rather than simply guessing. Guessing the answer is regarding as cheating and only 0.5 marks will be awarded.] (4 marks)

   Ans. $5257 = 2674 \times 1 + 2583$
   $2674 = 2583 \times 1 + 91$
   $2583 = 91 \times 28 + 35$
   $91 = 35 \times 2 + 21$
   $35 = 21 \times 1 + 14$
   $21 = 14 \times 1 + 7$
   $14 = 7 \times 2 + 0$ [1.5 marks]
   $7 = 21 - 14 \times 1$
   $7 = 21 - (35 - 21) = 21 \times 2 - 35$
   $7 = (91 - 35 \times 2) \times 2 - 35 = 91 \times 2 - 35 \times 5$
   $7 = 91 \times 2 - (2583 - 91 \times 28) \times 5 = 91 \times 142 - 2583 \times 5$
   $7 = (2674 - 2583) \times 142 - 2583 \times 5 = 2674 \times 142 - 2583 \times 147$
   $7 = 2674 \times 142 - (5257 - 2674) \times 147 = 2674 \times 289 - 5257 \times 147$ [2 marks]

   Therefore, there are $m = 289$ and $n = -147$ such that $2674m + 5257n = 7$. ....[0.5 mark]

(b) Use contrapositive proof to show that for all integers n, if $7n + 3$ is even then n is odd.

(2 marks)

Ans. Suppose n is an even integer. Then $n = 2k$ for some integer k. . . . . [0.5 mark]

$7n + 3 = 14k + 3 = 2(7k + 1) + 1 = 2k' + 1$. Here $k' = 7k + 1$ is some integer. . . . [1 mark]

By definition, $7n + 3$ is odd. . . . . [0.5 mark]

(c) Use the Principle of Mathematical Induction to prove the equality

$$2 + 3 + \cdots + n = \frac{(n-1)(n+2)}{2}$$

for integers $n \geq 2$. (3 marks)

Ans. Base Step:

When $n = 2$, RHS$=\frac{(2-1)(2+2)}{2} = \frac{4}{2} = 2 =$ LHS . . . . [1 mark]

Induction Step:

Assume $2 + 3 + \cdots + k = \frac{(k-1)(k+2)}{2}$ for $k \geq 2$. . . . . [0.3 mark]

$$\begin{aligned} 2 + 3 + \cdots + k + (k+1) &= \frac{(k-1)(k+2)}{2} + (k+1) && [0.5 \text{ mark}] \\ &= \frac{(k-1)(k+2) + 2(k+1)}{2} = \frac{k^2 + k - 2 + 2k + 2}{2} && [0.5 \text{ mark}] \\ &= \frac{k^2 + 3k}{2} = \frac{k(k+3)}{2} = \frac{((k+1)-1)((k+1)+2)}{2} && [0.5 \text{ mark}] \end{aligned}$$

Hence, by the Principle of Mathematical Induction, $2 + 3 + \cdots + n = \frac{(n-1)(n+2)}{2}$ for $n \geq 2$.

. . . . [0.2 mark]

(d) Use the Principle of Mathematical Induction to prove that the inequality

$$\prod_{i=1}^{n}\left(1-\frac{1}{3^i}\right) \geq \frac{1}{9}+\frac{1}{3^{n+1}}$$

for integers $n \geq 1$. Here $\prod_{i=1}^{n} f(i) = f(1)\times f(2)\times\cdots\times f(n)$ is the product of n terms for some expression f. (3 marks)

Ans. <u>Base Step:</u>

When $n=1$, LHS$=1-\frac{1}{3}=\frac{2}{3}\geq\frac{1}{9}+\frac{1}{9}=\frac{2}{9}$=RHS .... [1 mark]

<u>Induction Step:</u>

Assume $\prod_{i=1}^{k}\left(1-\frac{1}{3^i}\right)\geq\frac{1}{9}+\frac{1}{3^{k+1}}$ for $k\geq 1$. .... [0.3 mark]

$$\prod_{i=1}^{k+1}\left(1-\frac{1}{3^i}\right) = \left[\prod_{i=1}^{k}\left(1-\frac{1}{3^i}\right)\right]\left(1-\frac{1}{3^{k+1}}\right) \geq \left[\frac{1}{9}+\frac{1}{3^{k+1}}\right]\left(1-\frac{1}{3^{k+1}}\right) \qquad \text{[0.5 mark]}$$

$$= \frac{1}{9}+\frac{1}{3^{k+1}}-\frac{1}{3^{k+3}}-\frac{1}{3^{2k+2}} = \frac{1}{9}+\frac{3}{3^{k+2}}-\frac{\frac{1}{3}}{3^{k+2}}-\frac{\frac{1}{3^k}}{3^{k+2}} \qquad \text{[0.5 mark]}$$

$$\geq \frac{1}{9}+\frac{1}{3^{k+2}}\left[3-\frac{1}{3}-\frac{1}{3}\right] \geq \frac{1}{9}+\frac{1}{3^{k+2}} \qquad \text{[0.5 mark]}$$

Note that $k\geq 1 \Rightarrow 3^k\geq 3 \Rightarrow -\frac{1}{3^k}\geq -\frac{1}{3}$.

Hence, by the Principle of Mathematical Induction, $\prod_{i=1}^{n}\left(1-\frac{1}{3^i}\right)\geq\frac{1}{9}+\frac{1}{3^{n+1}}$ for $n\geq 1$.
.... [0.2 mark]

2. CO4. Express relations correctly with their mathematical properties. C2

(a) A set $A = \{a, b, c, d, e\}$ has a relation

$$R = \{(a,a),\ (a,c),\ (a,d),\ (b,b),\ (b,c),\ (b,e),\ (c,a),\ (c,c),\ (d,a),\ (d,b),\ (d,d), (e,a), (e,e)\}$$

Answer the following questions by filling the box with Yes or No.

i. Is R reflexive? .... Yes

ii. Is R symmetric? .... No

iii. Is R a partial order relation? .... No

Provide justifications for the above questions in the box below if your answer is No.

($3 \times 0.3 + 0.6$=1.5 mark)

Ans. ii. R is not symmetric because $(b, c) \in R$ but $(c, b) \notin R$. .... [0.3 mark]

iii. R is not a partial order because it is not antisymmetric since $(a, c), (c, a) \in R$ but $a \neq c$. .... [0.3 mark]

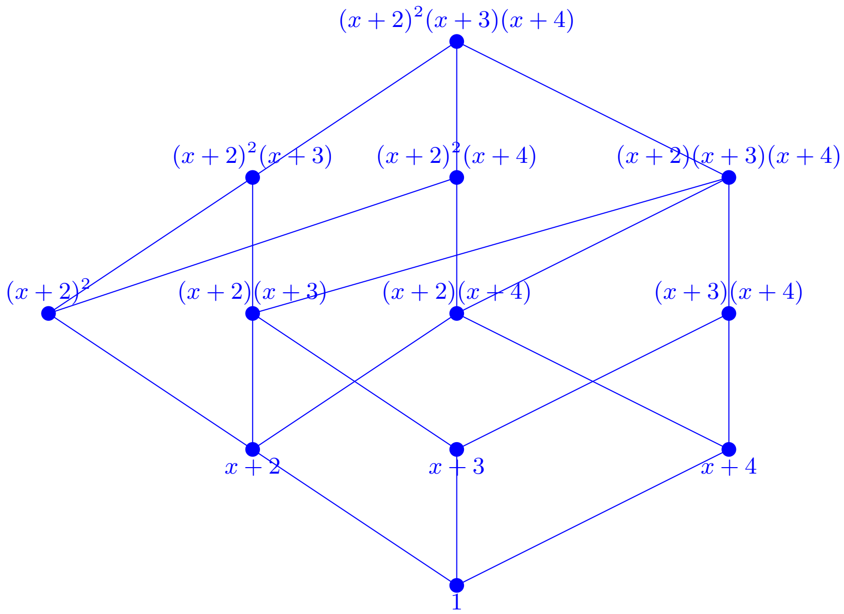
Note that the lecturer uses a symmetric pair for disclaiming anti-symmetry in lecture. This may not be precise enough.

Anti-symmetry: $\forall x \forall y (xRy \wedge yRx \to x = y)$

Not anti-symmetry: $\sim \big[\forall x \forall y (xRy \wedge yRx \to x = y)\big] \equiv \exists x \exists y \big[(xRy \wedge yRx \wedge x \neq y)\big]$

(b) Let A be the set of all the monic polynomial factors of the monic polynomial $(x+2)^2(x+3)(x+4)$ and the partial order relation $p \preceq q$ be p divides q for any monic polynomials p, q. Sketch the Hasse diagram of the poset $(A, \preceq)$. (1.5 marks)

Ans.



List out all elements of A in proper ordering .... [0.7 mark]

Correct links in the Hasse diagram .... [0.8 mark]

(c) Consider a set $A = \{a, b, c, d, e, f\}$ with the following relation

$$R = \{(a, c),\ (a, f),\ (b, c),\ (b, e),\ (c, b),\ (d, c),\ (e, d),\ (f, a)\}.$$

i. Write down the **matrix representation** of R and then (I) determine if the relation R is **irreflexive** with justification; (II) determine if the relation R is **anti-symmetric** with justifications. (2 marks)

Ans. $M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$ .... [1 mark]

(I) Since the diagonal of M_R are all zero, R is irreflexive. .... [0.5 mark]

(II) Since $(b, c), (c, b) \in R$ but $b \neq c$, R is **not anti-symmetric**. .... [0.5 mark]

ii. Apply the **Warshall algorithm** to find the matrix representation of the **transitive closure** of R. (3 marks)

Ans. Step 1: $\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & \boxed{1} & 0 & 0 & \boxed{1} \end{pmatrix} \end{array}$ .... [0.5 mark]

Step 2: $\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & \boxed{1} & 0 & \boxed{1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$ .... [0.5 mark]

Step 3: $\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & \boxed{1} & 1 & 0 & \boxed{1} & 1 \\ 0 & \boxed{1} & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & \boxed{1} & 1 & 0 & \boxed{1} & 1 \end{pmatrix} \end{array}$ .... [0.5 mark]

Step 4: $$\begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 0 & 1 & 1 & 0 & 1 & 1 \\ b & 0 & 1 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & 1 & 0 & 1 & 0 \\ d & 0 & 1 & 1 & 0 & 1 & 0 \\ e & 0 & \boxed{1} & \boxed{1} & 1 & \boxed{1} & 0 \\ f & 1 & 1 & 1 & 0 & 1 & 1 \end{array}$$ .... [0.5 mark]

Step 5: $$\begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 0 & 1 & 1 & \boxed{1} & 1 & 1 \\ b & 0 & 1 & 1 & \boxed{1} & 1 & 0 \\ c & 0 & 1 & 1 & \boxed{1} & 1 & 0 \\ d & 0 & 1 & 1 & \boxed{1} & 1 & 0 \\ e & 0 & 1 & 1 & 1 & 1 & 0 \\ f & 1 & 1 & 1 & \boxed{1} & 1 & 1 \end{array}$$ .... [0.5 mark]

Step 6: The matrix representation of the transitive closure is

$$\begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ b & 0 & 1 & 1 & 1 & 1 & 0 \\ c & 0 & 1 & 1 & 1 & 1 & 0 \\ d & 0 & 1 & 1 & 1 & 1 & 0 \\ e & 0 & 1 & 1 & 1 & 1 & 0 \\ f & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$ .... [0.5 mark]