# UECM1304/UECM1303 TEST 1 MARKING GUIDE

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COURSE CODE & COURSE TITLE: UECM1304/3 DISCRETE MATHEMATICS WITH APPLICATIONS
FACULTY: LKC FES, UTAR COURSE: AM, AS, SE
SESSION: MAY 2019 LECTURER: KOAY HANG LEEN, LIEW HOW HUI

**Instruction**: Answer all questions in the space provided. **If you do not write your answer in the space provided, you will get ZERO mark**. An answer without working steps may also receive ZERO mark.

1. CO1: Recognise statements and quantified statements. .................................C1

(a) Given the atomic statements $p$, $q$ and $r$. **State** the truth table of the following statement

$$\sim (p \vee (q \wedge \sim r)) \to (\sim p \wedge \sim q \wedge \sim r).$$

**Recognise** whether the statement is a tautology, contingency or contradiction (4.5 marks)

*Ans.* The truth table is stated below.

| $p$ | $q$ | $r$ | $\sim (p \vee (q \wedge \sim r)) \to (\sim p \wedge \sim q \wedge \sim r)$ |
|---|---|---|---|
| T | T | T | T |
| T | T | F | T |
| T | F | T | T |
| T | F | F | T |
| F | T | T | F |
| F | T | F | T |
| F | F | T | F |
| F | F | F | T |

................................................................ [0.5 × 8=4 marks]

Since there is a truth assignment in which the statement is true and there is a truth assignment in which the statement is false, the statement is a **contingency**. . . [0.5 mark]

(b) Given the domain of discourse is $\mathbb{R}$. Translate the following quantified statement

$$\forall x \exists y (x > 0 \to (y > 0 \wedge x = y^2))$$

to English sentence. Marks will be deducted if your English sentence is more than 18 words. (1 mark)

*Ans.* Every positive number is the square of some positive number. ............ [1 mark]

(c) Given that $p$, $q$, $r$ are atomic statements, for the statement

$$(p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r),$$

**identify** the logical equivalent statement with no more than 8 logical connectives from the set $\{\sim, \wedge, \vee\}$. If you use the logical connectives $\to$ and $\leftrightarrow$, marks will be deducted.

(2 marks)

*Ans.* $(p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

$\equiv (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge (r \vee \sim r)) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

[2,3 distributive law; 0.4 mark]

$\equiv (p \wedge q \wedge \sim r) \vee (p \wedge \sim q) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

[2, negation and identity; 0.4 mark]

$\equiv ((p \vee \sim p) \wedge q \wedge \sim r) \vee (p \wedge \sim q) \vee (\sim p \wedge q \wedge r)$ [1,4 distributive law; 0.3 mark]

$\equiv (q \wedge \sim r) \vee (p \wedge \sim q) \vee (\sim p \wedge q \wedge r)$ [1, negation and identity; 0.4 mark]

$\equiv (q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r) \vee (p \wedge \sim q) \vee (\sim p \wedge q \wedge r)$ [1, absorption law; 0.2 mark]

$\equiv (q \wedge \sim r) \vee (\sim p \wedge q) \vee (p \wedge \sim q)$ [2,4, distributive, negation, identity; 0.3 mark]

(d) Use the **laws of logical equivalence** to transform the following quantified statement

$$(\exists x \forall y (p(x,y))) \vee \sim \exists y (q(y) \to \forall z r(z))$$

to prenex normal form. (2 marks)

*Ans.* $(\exists x \forall y (p(x,y))) \vee \sim \exists y (q(y) \to \forall z r(z))$

$\equiv \quad (\exists x \forall y (p(x,y))) \vee \forall y \sim (q(y) \wedge \sim \forall z r(z))$ Generalised De Morgan [0.3 mark]

$\equiv \quad (\exists x \forall y (p(x,y))) \vee \forall y (q(y) \wedge \sim \forall z r(z))$ Implication & De Morgan [0.4 mark]

$\equiv \quad (\exists x \forall y (p(x,y))) \vee \forall y (q(y) \wedge \exists z \sim r(z))$ Generalised De Morgan [0.3 mark]

$\equiv \quad (\exists x \forall y (p(x,y))) \vee \forall y \exists z (q(y) \wedge \sim r(z))$ Free variable law [0.4 mark]

$\equiv \quad \exists x \forall y (p(x,y) \vee \forall y_2 \exists z (q(y_2) \wedge \sim r(z)))$ Free variable law [0.3 mark]

$\equiv \quad \exists x \forall y \forall y_2 \exists z (p(x,y) \vee (q(y_2) \wedge \sim r(z)))$ Free variable law [0.3 mark]

(e) The definition of "pointwise convergence" of a sequence of functions $\{f_n\}$ to a function $f$ on an interval $A \subset \mathbb{R}$ can be defined by the following quantified statement

$$\forall x \forall \epsilon \exists N \forall n \Big[ (x \in A) \to \Big[ (\epsilon > 0) \to \Big( (N \in \mathbb{N}) \wedge ((n \geq N) \to |f_n(x) - f(x)| < \epsilon) \Big) \Big] \Big].$$

Write the negation of this statement in prenex normal form, i.e. apply $\sim$ to the quantified statement and then write it into the logically equivalent prenex normal form. (0.5 mark)

*Ans.* The negation of the quantified statement is

$$\exists x \exists \epsilon \forall N \exists n \Big[ (x \in A) \wedge \Big[ (\epsilon > 0) \wedge \Big( (N \in \mathbb{N}) \to ((n \geq N) \wedge |f_n(x) - f(x)| \geq \epsilon) \Big) \Big] \Big].$$ [0.5 mark]

2. CO2. Determine the validity of an argument. .......................................... C2

(a) Given the following argument:

$$\begin{array}{rr} & p \vee q \\ & p \wedge q \to r \\ & q \wedge \sim r \\ \hline \therefore & \sim p \end{array}$$

Either use the comparison table to **defend** that the argument is valid or **give a counter example** to show that the argument is invalid. (4 marks)

*Ans.* The comparison table is stated below: ................................. [3.5 marks]

| $p$ | $q$ | $r$ | $p \vee q$ | $p \wedge q \to r$ | $q \wedge \sim r$ | $\sim p$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F |
| T | T | F | T | F | T | F |
| T | F | T | T | T | F | F |
| T | F | F | T | T | F | F |
| F | T | T | T | T | F | T |
| F | T | F | T | T | T | T |
| F | F | T | F | T | F | T |
| F | F | F | F | T | F | T |

Scanning through rows 1 to 4 of the comparison table, there is no truth assignment for which the premises are true but the conclusion is false. Therefore the argument is valid.
........................................................................ [0.5 mark]

(b) Let $p$, $q$, $r$ and $s$ be atomic statements. Use the **laws of logical equivalence and implication** to *explain* the validity of the following argument:

$$\begin{array}{rr} & \sim p \vee q \\ & \sim q \vee s \\ \hline \therefore & \sim p \vee r \vee s \end{array}$$

(2 marks)

*Ans.*

| | | |
|---|---|---|
| $\phi_1$ | $\sim p \vee q$ | premise |
| $\phi_2$ | $\sim q \vee s$ | premise |
| $\psi_1$ | $p \to q$ | $\phi_1$, Implication law ................. [0.4 mark] |
| $\psi_2$ | $q \to s$ | $\phi_2$, Implication law ................. [0.4 mark] |
| $\psi_3$ | $p \to s$ | $\psi_1, \psi_2$, Transitivity ................. [0.4 mark] |
| $\psi_4$ | $\sim p \vee s$ | $\psi_3$, Implication law ................. [0.4 mark] |
| $\therefore$ | $\sim p \vee r \vee s$ | $\psi_4$, Generalisation, Commutative law [0.4 mark] |

(c) Use **only** the **rules of inference** and fitch style proof to **infer** the argument

$$p \to \sim (q \vee r),\ q \vdash \sim p.$$

[**Warning:** If you use any other rules, you will receive ZERO for this question.] (2 marks)

*Ans.*

| | | |
|---|---|---|
| 1 | $p \to \sim (q \vee r)$ | premise |
| 2 | $q$ | premise |
| 3 | $p$ | assumption .......................... [0.4 mark] |
| 4 | $\sim (q \vee r)$ | 1,3 $\to$E .......................... [0.4 mark] |
| 5 | $q \vee r$ | 2 $\vee$I .......................... [0.4 mark] |
| 6 | $\perp$ | 4,5 $\neg$E .......................... [0.4 mark] |
| 7 | $\sim p$ | 3–6 $\neg$I .......................... [0.4 mark] |

(d) Let $P(x)$ and $Q(x)$ be predicates. Use **either** the *laws of logical equivalence and implication* **or** *rules of inference* to *explain* the validity of the following argument:

$$\forall x(Q(x) \to P(x)),\ \exists x Q(x) / \therefore \exists x(Q(x) \wedge P(x)).$$ (2 marks)

*Ans.* Using the laws of logical equivalence and implication, the inference goes as follows:

| | | |
|---|---|---|
| $\phi_1$ | $\forall x(Q(x) \to P(x))$ | premise |
| $\phi_2$ | $\exists x Q(x)$ | premise |
| $\psi_1$ | $Q(s)$ | $\phi_2$ existential initialisation ............... [0.4 mark] |
| $\psi_2$ | $Q(s) \to P(s)$ | $\phi_1$ universal initialisation ................. [0.4 mark] |
| $\psi_3$ | $P(s)$ | $\psi_1, \psi_2$ MP ................................ [0.4 mark] |
| $\psi_4$ | $Q(s) \wedge P(s)$ | $\psi_1, \psi_3$ conjuction ........................ [0.4 mark] |
| $\therefore$ | $\exists x(Q(x) \wedge P(x))$ | $\psi_4$ existential generalisation ............. [0.4 mark] |

**Laws of Logical Equivalence and Implication**

Let $p$, $q$ and $r$ be atomic statements, $T$ be a tautology and $F$ be a contradiction. Suppose the variable $x$ has no free occurrences in $\xi$ and is substitutable for $x$ in $\xi$. Then

1. Double negative law: $\sim(\sim p) \equiv p$.
2. Idempotent laws: $p \wedge p \equiv p$; $p \vee p \equiv p$.
3. Universal bound laws: $p \vee T \equiv T$; $p \wedge F \equiv F$.
4. Identity laws: $p \wedge T \equiv p$; $p \vee F \equiv p$.
5. Negation laws: $p \vee \sim p \equiv T$; $p \wedge \sim p \equiv F$.
6. Commutative laws: $p \wedge q \equiv q \wedge p$; $p \vee q \equiv q \vee p$.
7. Absorption laws: $p \vee (p \wedge q) \equiv p$; $p \wedge (p \vee q) \equiv p$.
8. Associative laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$; $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
9. Distributive laws: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$;
   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
10. De Morgan's laws: $\sim(p \wedge q) \equiv \sim p \vee \sim q$; $\sim(p \vee q) \equiv \sim p \wedge \sim q$.
11. Implication law: $p \to q \equiv \sim p \vee q$
12. Biconditional law: $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.
13. Modus Ponens (MP in short): $p \to q,\ p \models q$
14. Modus Tollens (MT in short): $p \to q,\ \sim q \models \sim p$
15. Generalisation: $p \models p \vee q$; $q \models p \vee q$
16. Specialisation: $p \wedge q \models p$; $p \wedge q \models q$
17. Conjunction: $p,\ q \models p \wedge q$
18. Elimination: $p \vee q,\ \sim q \models p$; $p \vee q,\ \sim p \models q$
19. Transitivity: $p \to q,\ q \to r \models p \to r$
20. Contradiction Rule: $\sim p \to F \models p$
21. Quantified de Morgan laws: $\sim \forall x\phi \equiv \exists x \sim \phi$; $\sim \exists x\phi \equiv \forall x \sim \phi$;
22. Quantified conjunctive law: $\forall x(\phi \wedge \psi) \equiv (\forall x\ \phi) \wedge (\forall x\ \psi)$;
23. Quantified disjunctive law: $\exists x(\phi \vee \psi) \equiv (\exists x\ \phi) \vee (\exists x\ \psi)$;
24. Quantifiers swapping laws: $\forall x\forall y\phi \equiv \forall y\forall x\phi$; $\exists x\exists y\phi \equiv \exists y\exists x\phi$;
25. Independent quantifier law: $\xi \equiv \forall x\xi \equiv \exists x\xi$;
26. Variable renaming laws: $\forall x\phi \equiv \forall y\phi[y/x]$; $\exists x\phi \equiv \exists y\phi[y/x]$;
27. Free variable laws: $\forall x(\xi \wedge \psi) \equiv \xi \wedge (\forall x\psi)$; $\exists x(\xi \wedge \psi) \equiv \xi \wedge (\exists x\psi)$;
    $\forall x(\xi \vee \psi) \equiv \xi \vee (\forall x\psi)$; $\exists x(\xi \vee \psi) \equiv \xi \vee (\exists x\psi)$;
28. Universal instantiation: $\forall x\phi \Rightarrow \phi[a/x]$;
29. Universal generalisation: $\phi[a/x] \Rightarrow \forall x\phi$;
30. Existential instantiation: $\exists x\phi \Rightarrow \phi[s/x]$;
31. Existential generalisation: $\phi[s/x] \Rightarrow \exists x\phi$.

**Rules of Inference**

Let $\phi$, $\psi$, $\xi$ be any well-formed formulae. Then

1. $\wedge$-introduction: $\phi,\ \psi \vdash \phi \wedge \psi$
2. $\wedge$-elimination: $\phi \wedge \psi \vdash \phi$ or $\phi \wedge \psi \vdash \psi$
3. $\rightarrow$-introduction: $\boxed{\phi,\ \cdots,\ \psi} \vdash (\phi \rightarrow \psi)$
4. $\rightarrow$-elimination: $\phi \rightarrow \psi,\ \phi \vdash \psi$
5. $\vee$-introduction: $\phi \vdash \phi \vee \psi$ or $\psi \vdash \phi \vee \psi$
6. $\vee$-elimination: $\phi \vee \psi,\ \boxed{\phi,\ \cdots,\ \xi},\ \boxed{\psi,\ \cdots,\ \xi} \vdash \xi$
7. $\neg$-introduction or $\sim$-introduction: $\boxed{\sim \phi,\ \cdots,\ \bot} \vdash \phi$ or $\boxed{\phi,\ \cdots,\ \bot} \vdash \sim \phi$
8. $\neg$-elimination or $\sim$-elimination: $\phi,\ \sim \phi \vdash \bot$
9. $\forall$-introduction: $\phi(a) \vdash \forall x \phi(x)$
10. $\forall$-elimination: $\forall x \phi(x) \vdash \phi(t)$
11. $\exists$-introduction: $\phi(t) \vdash \exists x \phi(x)$
12. $\exists$-elimination: $\exists x \phi(x), \boxed{\phi(s) \cdots \xi} \vdash \xi$

The term $t$ is free with respect to $x$ in $\phi$ and $[t/x]$ means "$t$ replaces $x$".