UECM1304 Test 1 Marking Guide

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Course Code & Course Title:	UECM1304 Discrete Mathematics with Applications		
Faculty:	LKC FES, UTAR	Course:	AM
Trimester:	Jun 2025	Lecturer:	Liew How Hui

Instruction: Answer all questions in the space provided. **If you do not write your answer in the space provided, you will get ZERO mark**. An answer without working steps may also receive ZERO mark.

1. CO1: Recognise statements and quantified statements. C1

   (a) Given the atomic statements p, q and r. **State** the truth table of the following statement

$$\underbrace{(p \leftrightarrow (q \vee r)) \wedge ((\sim p \to q) \to (q \wedge r))}_{\phi}.$$

State if the statement ϕ is a tautology, contingency or contradiction. If the statement ϕ is satisfiable, write down a truth assignment for which $v(\phi) = \text{T}$. (5 marks)

Ans. The truth table is stated below.

p	q	r	$p \leftrightarrow (q \vee r)$	$(\sim p \to q) \to (q \wedge r)$	$(p \leftrightarrow (q \vee r)) \wedge ((\sim p \to q) \to (q \wedge r))$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	F	F
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	F	T	F
F	F	F	T	T	T

.... [0.5 × 8=4 marks]

Since there is a truth assignment in which the statement is true and there is a truth assignment in which the statement is false, the statement is a **contingency**. .. [0.5 mark]

A true assignment for which $v(\phi) = \text{T}$ is $v(p) = v(q) = v(r) = \text{T}$. .... [0.5 mark]

(b) Given that p, q, r, s are atomic statements, prove the following logical equivalence

$$(p \wedge q \wedge r \rightarrow s) \wedge (r \rightarrow p \vee q \vee s) \equiv ((p \leftrightarrow q) \wedge r) \rightarrow s$$

by writing down the step-by-step laws of logical equivalences used. (2.5 marks)

Ans. $(p \wedge q \wedge r \rightarrow s) \wedge (r \rightarrow p \vee q \vee s)$

$\equiv (\sim (p \wedge q \wedge r) \vee s) \wedge (\sim r \vee p \vee q \vee s)$ [implication law; 0.4 mark]

$\equiv (\sim p \vee \sim q \vee \sim r \vee s) \wedge (\sim r \vee p \vee q \vee s)$ [De Morgan law; 0.2 mark]

$\equiv ((\sim p \vee \sim q) \wedge (p \vee q)) \vee (\sim r \vee s)$ [distributive law (also assoc, comm laws); 0.4 mark]

$\equiv \sim (((\sim p \vee \sim q) \wedge (p \vee q)) \vee \sim r) \rightarrow s$

[Associative, Implication, Double Negative; 0.2 mark]

$\equiv (((p \wedge q) \vee (\sim p \wedge \sim q)) \wedge r) \rightarrow s$ [De Morgan, Double Negative; 0.4 mark]

$\equiv (((p \vee \sim p) \wedge (p \vee \sim q) \wedge (q \vee \sim p) \wedge (q \vee \sim q)) \wedge r) \rightarrow s$ [Distributive; 0.2 mark]

$\equiv ((\mathrm{T} \wedge (p \vee \sim q) \wedge (q \vee \sim p) \wedge \mathrm{T}) \wedge r) \rightarrow s$ [Negation law; 0.2 mark]

$\equiv (((p \vee \sim q) \wedge (q \vee \sim p)) \wedge r) \rightarrow s$ [Identity law; 0.2 mark]

$\equiv (((p \rightarrow q) \wedge (q \rightarrow p)) \wedge r) \rightarrow s$ [Implication law, Commutative law; 0.2 mark]

$\equiv ((p \leftrightarrow q) \wedge r) \rightarrow s$ [Biconditional law; 0.1 mark]

(c) Let $P(x)$ and $Q(x,y)$ be predicates. Transform the following quantified statement

$$\forall x(P(x) \to Q(x,y)) \to (\exists y P(y) \wedge \exists z Q(y,z))$$

to an equivalent **prenex normal form**. You need to state the laws used in the derivation of the prenex normal form. (2 marks)

Ans. $\forall x(P(x) \to Q(x,y)) \to (\exists y P(y) \wedge \exists z Q(y,z))$

$\equiv \quad \forall x(P(x) \to Q(x,y)) \to (\exists y_2 P(y_2) \wedge \exists z Q(y,z))$ Variable Renaming [0.4 mark]

$\equiv \quad \sim \forall x(P(x) \to Q(x,y)) \vee (\exists y_2 P(y_2) \wedge \exists z Q(y,z))$ Implication [0.3 mark]

$\equiv \quad \exists x \sim (P(x) \to Q(x,y)) \vee (\exists y_2 P(y_2) \wedge \exists z Q(y,z))$ Generalised De Morgan [0.3 mark]

$\equiv \quad \exists x \sim (P(x) \to Q(x,y)) \vee \exists y_2 \exists z \Big(P(y_2) \wedge Q(y,z)\Big)$ Free variable law [0.4 mark]

$\equiv \quad \exists x \Big[\sim (P(x) \to Q(x,y)) \vee \exists y_2 \exists z \Big(P(y_2) \wedge Q(y,z)\Big)\Big]$ Free variable law [0.2 mark]

$\equiv \quad \exists x \exists y_2 \exists z \Big[\sim (P(x) \to Q(x,y)) \vee \Big(P(y_2) \wedge Q(y,z)\Big)\Big]$ Free variable law [0.4 mark]

(d) The definition of "uniform convergence" of a sequence of functions $\{f_n\}$ to a function f on an interval $A \subset \mathbb{R}$ can be defined by the following quantified statement

$$\forall \epsilon \exists N \forall n \forall x \Big[(\epsilon > 0) \to \Big((N \in \mathbb{N}) \wedge ((n \geq N) \to (x \in A) \to |f_n(x) - f(x)| < \epsilon)\Big)\Big].$$

Write the negation of this statement in prenex normal form, i.e. apply $\sim$ to the quantified statement and then write it into the logically equivalent prenex normal form. You do not need to write down the laws for this question. (0.5 mark)

Ans. The negation of the quantified statement is

$$\exists \epsilon \forall N \exists n \exists x \Big[(\epsilon > 0) \wedge \Big((N \in \mathbb{N}) \to ((n \geq N) \wedge (x \in A) \wedge |f_n(x) - f(x)| \geq \epsilon)\Big)\Big].$$ [0.5 mark]

2. CO2. Determine the validity of an argument. C2

(a) Given the following argument:

$$\begin{array}{r} \sim p \vee q \\ r \to \sim q \\ \hline \therefore \ \ p \to (q \wedge \sim r) \end{array}$$

Determine whether the argument is valid by using comparison table method or truth table method. (4 marks)

Ans. The comparison table is stated below: .... [3.5 marks]

p	q	r	$\sim p \vee q$	$r \to \sim q$	$p \to q \wedge \sim r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Looking at row 1, row 3 and row 4 of the comparison table, there is no truth assignment for which the premises are all true but the conclusion is false. Therefore the argument is valid. .... [0.5 mark]

(b) Let p, q, r, s, u be atomic statements. Show that

$$p \vee q \to r \wedge s, \ s \vee t \to u \ \models \ p \to u$$

using laws of logical equivalences and implications or tableaux method. In either method, the **essential laws** that you deployed need to be stated. (2 marks)

Ans. A sample proof using laws of logical equivalences and implications is

ϕ_1 :	$p \vee q \to r \wedge s$	premise
ϕ_2 :	$s \vee t \to u$	premise
ψ_1 :	$(\sim p \wedge \sim q) \vee (r \wedge s)$	ϕ_1, Implication, DM
ψ_2 :	$(\sim p \vee r) \wedge (\sim p \vee s) \wedge (\sim q \vee r) \wedge (\sim q \vee s)$	ψ_1, Distribution
ψ_3 :	$\sim p \vee s$	ψ_2, Specialisation
ψ_4 :	$(\sim s \wedge \sim t) \vee u$	ϕ_2, Implication, DM
ψ_5 :	$(\sim s \vee u) \wedge (\sim t \vee u)$	ψ_4, Distribution
ψ_6 :	$\sim s \vee u$	ψ_5, Specialisation
ψ_7	$\sim p \vee u$	ψ_3, ψ_6, Resolution
$\therefore$	$p \to u$	ψ_7, Implication

Premises .... [0.2 marks]

Essential steps .... [1.8 marks]

(c) Use **only** the **rules of inference** and fitch style proof to **infer** the validity of the argument

$$p \to q,\ r \to s,\ p \vee r \vdash q \vee s.$$

[**Warning:** If you use any rules other than the rules of inference, you will receive ZERO for this question.] (2 marks)

Ans.

1	$p \to q$	premise	
2	$r \to s$	premise	
3	$p \vee r$	premise	
4	$\quad p$	assumption	[0.2 mark]
5	$\quad q$	1,4 $\to$E	[0.4 mark]
6	$\quad q \vee s$	5 $\vee I_1$	[0.3 mark]
7	$\quad r$	assumption	[0.2 mark]
8	$\quad s$	1,4 $\to$E	[0.4 mark]
9	$\quad q \vee s$	5 $\vee I_2$	[0.2 mark]
10	$q \vee s$	3, 4–6, 7–9 $\vee$E	[0.3 mark]

(d) Let $R(x)$, $Q(x)$ and $I(x)$ be predicates with single arity. Use **either** the *laws of logical equivalence and implication* **or** *rules of inference* to show that the following argument is valid:

$$\forall x(Q(x) \to R(x)),\ \exists x(Q(x) \wedge I(x))\ /\ \therefore \exists x(R(x) \wedge I(x)).$$ (2 marks)

Ans. Using the laws of logical equivalence and implication, the inference goes as follows:

ϕ_1	$\forall x(Q(x) \to R(x))$	premise	
ϕ_2	$\exists x(Q(x) \wedge I(x))$	premise	
ψ_1	$Q(s) \wedge I(s)$	ϕ_2 existential initialisation	[0.4 mark]
ψ_2	$Q(s) \to R(s)$	ϕ_1 universal initialisation	[0.4 mark]
ψ_3	$Q(s)$	ψ_1 Specialisation	[0.3 mark]
ψ_4	$R(s)$	ψ_2, ψ_3 MP	[0.3 mark]
ψ_5	$I(s)$	ψ_1 Specialisation	[0.2 mark]
ψ_6	$R(s) \wedge I(s)$	ψ_4, ψ_5 conjuction	[0.2 mark]
$\therefore$	$\exists x(R(x) \wedge I(x))$	ψ_6 existential generalisation	[0.2 mark]

Laws of Logical Equivalences and Implications

Let p, q and r be atomic statements, T be a tautology and F be a contradiction. Suppose the variable x has no free occurrences in ξ and is substitutable for x in ξ. Then

1.	Double negative law:	$\sim(\sim p) \equiv p$.
2.	Idempotent laws:	$p \wedge p \equiv p$; $\quad p \vee p \equiv p$.
3.	Universal bound laws:	$p \vee T \equiv T$; $\quad p \wedge F \equiv F$.
4.	Identity laws:	$p \wedge T \equiv p$; $\quad p \vee F \equiv p$.
5.	Negation laws:	$p \vee \sim p \equiv T$; $\quad p \wedge \sim p \equiv F$.
6.	Commutative laws:	$p \wedge q \equiv q \wedge p$; $\quad p \vee q \equiv q \vee p$.
7.	Absorption laws:	$p \vee (p \wedge q) \equiv p$; $\quad p \wedge (p \vee q) \equiv p$.
8.	Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$; $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
9.	Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$; $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
10.	De Morgan's laws:	$\sim(p \wedge q) \equiv \sim p \vee \sim q$; $\sim(p \vee q) \equiv \sim p \wedge \sim q$.
11.	Implication law:	$p \to q \equiv \sim p \vee q$
12.	Contrapositive law:	$p \to q \equiv \sim q \to \sim p$
13.	Biconditional law:	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.
14.	Contradiction Rule:	$\sim p \to F \models p$
15.	Conjunction:	$p,\ q \models p \wedge q$
16.	Specialisation:	$p \wedge q \models p$; $\quad p \wedge q \models q$
17.	Generalisation:	$p \models p \vee q$; $\quad q \models p \vee q$
18.	Elimination:	$p \vee q,\ \sim q \models p$; $\quad p \vee q,\ \sim p \models q$
19.	Modus Ponens (MP in short):	$p \to q,\ p \models q$
20.	Modus Tollens (MT in short):	$p \to q,\ \sim q \models \sim p$
21.	Transitivity:	$p \to q,\ q \to r \models p \to r$
22.	Resolution:	$p \vee r,\ q \vee \sim r \models p \vee q$
23.	Quantified de Morgan laws:	$\sim \forall x\phi \equiv \exists x \sim \phi$; $\sim \exists x\phi \equiv \forall x \sim \phi$;
24.	Quantified conjunctive law:	$\forall x(\phi \wedge \psi) \equiv (\forall x\ \phi) \wedge (\forall x\ \psi)$;
25.	Quantified disjunctive law:	$\exists x(\phi \vee \psi) \equiv (\exists x\ \phi) \vee (\exists x\ \psi)$;
26.	Universal quantifiers swapping law:	$\forall x \forall y \phi \equiv \forall y \forall x \phi$;
27.	Existential quantifiers swapping law:	$\exists x \exists y \phi \equiv \exists y \exists x \phi$;
28.	Independent quantifier law:	$\xi \equiv \forall x \xi \equiv \exists x \xi$;
29.	Variable renaming laws:	$\forall x\phi \equiv \forall y \phi[y/x]$; $\quad \exists x\phi \equiv \exists y \phi[y/x]$;
30.	Free variable laws:	$\forall x(\xi \wedge \psi) \equiv \xi \wedge (\forall x\psi)$; $\quad \exists x(\xi \wedge \psi) \equiv \xi \wedge (\exists x\psi)$; $\forall x(\xi \vee \psi) \equiv \xi \vee (\forall x\psi)$; $\quad \exists x(\xi \vee \psi) \equiv \xi \vee (\exists x\psi)$;
31.	Universal instantiation:	$\forall x\phi \Rightarrow \phi[a/x]$;

32. Universal generalisation: $\phi[a/x] \Rightarrow \forall x \phi$;

33. Existential instantiation: $\exists x \phi \Rightarrow \phi[s/x]$;

34. Existential generalisation: $\phi[s/x] \Rightarrow \exists x \phi$.

Rules of Inference

Let ϕ, ψ, ξ be any well-formed formulae. Then

1. $\wedge$-introduction: $\phi,\ \psi \vdash \phi \wedge \psi$
2. $\wedge$-elimination: $\phi \wedge \psi \vdash \phi$ or $\phi \wedge \psi \vdash \psi$
3. $\rightarrow$-introduction: $\boxed{\phi,\ \cdots,\ \psi} \vdash (\phi \rightarrow \psi)$
4. $\rightarrow$-elimination: $\phi \rightarrow \psi,\ \phi \vdash \psi$
5. $\vee$-introduction: $\phi \vdash \phi \vee \psi$ or $\psi \vdash \phi \vee \psi$
6. $\vee$-elimination: $\phi \vee \psi,\ \boxed{\phi,\ \cdots,\ \xi},\ \boxed{\psi,\ \cdots,\ \xi} \vdash \xi$
7. $\neg$-introduction or $\sim$-introduction: $\boxed{\sim \phi,\ \cdots,\ \bot} \vdash \phi$ or $\boxed{\phi,\ \cdots,\ \bot} \vdash \sim \phi$
8. $\neg$-elimination or $\sim$-elimination: $\phi,\ \sim \phi \vdash \bot$
9. $\bot$-elimination: $\bot \vdash \phi$
10. $\forall$-introduction: ${}^t\phi(t) \vdash \forall x \phi(x)$
11. $\forall$-elimination: $\forall x \phi(x) \vdash \phi(t)$
12. $\exists$-introduction: $\phi(s) \vdash \exists x \phi(x)$
13. $\exists$-elimination: $\exists x \phi(x), \boxed{\phi(s) \cdots \xi} \vdash \xi$

The term t is free with respect to x in ϕ and $[t/x]$ means "t replaces x".