Laws of Logical Equivalences and Implications

Let p, q and r be atomic statements, T be a tautology and F be a contradiction. Suppose the variable x has no free occurrences in ξ and is substitutable for x in ξ. Then

1. Double negative law: $\sim(\sim p) \equiv p$.
2. Idempotent laws: $p \wedge p \equiv p$; $p \vee p \equiv p$.
3. Universal bound laws: $p \vee T \equiv T$; $p \wedge F \equiv F$.
4. Identity laws: $p \wedge T \equiv p$; $p \vee F \equiv p$.
5. Negation laws: $p \vee \sim p \equiv T$; $p \wedge \sim p \equiv F$.
6. Commutative laws: $p \wedge q \equiv q \wedge p$; $p \vee q \equiv q \vee p$.
7. Absorption laws: $p \vee (p \wedge q) \equiv p$; $p \wedge (p \vee q) \equiv p$.
8. Associative laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$;
   $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
9. Distributive laws: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$;
   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
10. De Morgan's laws: $\sim(p \wedge q) \equiv \sim p \vee \sim q$;
    $\sim(p \vee q) \equiv \sim p \wedge \sim q$.
11. Implication law: $p \to q \equiv \sim p \vee q$
12. Biconditional law: $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.
13. Modus Ponens (MP in short): $p \to q,\ p \models q$
14. Modus Tollens (MT in short): $p \to q,\ \sim q \models \sim p$
15. Generalisation: $p \models p \vee q$; $q \models p \vee q$
16. Specialisation: $p \wedge q \models p$; $p \wedge q \models q$
17. Conjunction: $p,\ q \models p \wedge q$
18. Elimination: $p \vee q,\ \sim q \models p$; $p \vee q,\ \sim p \models q$
19. Transitivity: $p \to q,\ q \to r \models p \to r$
20. Contradiction Rule: $\sim p \to F \models p$
21. Quantified de Morgan laws: $\sim \forall x \phi \equiv \exists x \sim \phi$; $\sim \exists x \phi \equiv \forall x \sim \phi$;
22. Quantified conjunctive law: $\forall x(\phi \wedge \psi) \equiv (\forall x\ \phi) \wedge (\forall x\ \psi)$;
23. Quantified disjunctive law: $\exists x(\phi \vee \psi) \equiv (\exists x\ \phi) \vee (\exists x\ \psi)$;
24. Universal quantifiers swapping law: $\forall x \forall y \phi \equiv \forall y \forall x \phi$;
25. Existential quantifiers swapping law: $\exists x \exists y \phi \equiv \exists y \exists x \phi$;
26. Independent quantifier law: $\xi \equiv \forall x \xi \equiv \exists x \xi$;
27. Variable renaming laws: $\forall x \phi \equiv \forall y \phi[y/x]$; $\exists x \phi \equiv \exists y \phi[y/x]$;

28. Free variable laws: $\forall x(\xi \wedge \psi) \equiv \xi \wedge (\forall x\psi)$; $\exists x(\xi \wedge \psi) \equiv \xi \wedge (\exists x\psi)$;
$\forall x(\xi \vee \psi) \equiv \xi \vee (\forall x\psi)$; $\exists x(\xi \vee \psi) \equiv \xi \vee (\exists x\psi)$;

29. Universal instantiation: $\forall x\phi \Rightarrow \phi[a/x]$;

30. Universal generalisation: $\phi[a/x] \Rightarrow \forall x\phi$;

31. Existential instantiation: $\exists x\phi \Rightarrow \phi[s/x]$;

32. Existential generalisation: $\phi[s/x] \Rightarrow \exists x\phi$.

Rules of Inference

Let ϕ, ψ, ξ be any well-formed formulae. Then

1. $\wedge$-introduction: $\phi,\ \psi \vdash \phi \wedge \psi$
2. $\wedge$-elimination: $\phi \wedge \psi \vdash \phi$ or $\phi \wedge \psi \vdash \psi$
3. $\rightarrow$-introduction: $\boxed{\phi,\ \cdots,\ \psi} \vdash (\phi \rightarrow \psi)$
4. $\rightarrow$-elimination: $\phi \rightarrow \psi,\ \phi \vdash \psi$
5. $\vee$-introduction: $\phi \vdash \phi \vee \psi$ or $\psi \vdash \phi \vee \psi$
6. $\vee$-elimination: $\phi \vee \psi,\ \boxed{\phi,\ \cdots,\ \xi},\ \boxed{\psi,\ \cdots,\ \xi} \vdash \xi$
7. $\neg$-introduction or $\sim$-introduction: $\boxed{\sim \phi,\ \cdots,\ \bot} \vdash \phi$ or $\boxed{\phi,\ \cdots,\ \bot} \vdash \sim \phi$
8. $\neg$-elimination or $\sim$-elimination: $\phi,\ \sim \phi \vdash \bot$
9. $\bot$-elimination: $\bot \vdash \phi$
10. $\forall$-introduction: ${}^{t}\phi(t) \vdash \forall x \phi(x)$
11. $\forall$-elimination: $\forall x \phi(x) \vdash \phi(t)$
12. $\exists$-introduction: $\phi(s) \vdash \exists x \phi(x)$
13. $\exists$-elimination: $\exists x \phi(x), \boxed{\phi(s) \cdots \xi} \vdash \xi$

The term t is free with respect to x in ϕ and $[t/x]$ means "t replaces x".