UNIVERSITI TUNKU ABDUL RAHMAN

ACADEMIC YEAR 2019/2020

SEPTEMBER EXAMINATION

UECM1304/UECM1303 DISCRETE MATHEMATICS WITH APPLICATIONS

WEDNESDAY, 11 SEPTEMBER 2019 TIME : 9.00 AM – 11.00 AM (2 HOURS)

BACHELOR OF SCIENCE (HONS) APPLIED MATHEMATICS WITH COMPUTING
BACHELOR OF SCIENCE (HONS) ACTUARIAL SCIENCE
BACHELOR OF SCIENCE (HONS) SOFTWARE ENGINEERING

Instruction to Candidates :

- Answer **ONE** question in PART A.
- Answer **ALL** questions in PART B.
- All questions carry equal marks.
- Write ALL your steps, marks will NOT be awarded if steps are skipped!

PART A: Answer **ONE** question only.

Q1. (a) Let p, q, r be atomic statements. **State** the truth table for the following compound statement

$$\sim (p \to ((p \vee q) \wedge r)).$$

Use the truth table to **recognise** whether the compound statement is a tautology, contingency or contradiction. (10 marks)

(b) Show that the statement $(p \to q \vee r)$ and the statement $(p \wedge q \to r)$ are not logically equivalent. (4 marks)

(c) Simplify the following statement

$$((p \vee q) \to (p \wedge q)) \vee (\sim p \wedge q).$$

to a logically equivalent statement with no more than TWO(2) logical connectives from the set $\{\sim, \wedge, \vee\}$ by stating the law used in each step of the simplification. (5 marks)

(d) Let $F(u,x,y)$, $G(y,v)$ and $H(x)$ be predicates. **List** down the steps and the logical equivalence rules to transform the following quantified statement

$$\sim \left[\forall x \exists y F(u,x,y) \to \exists x \left(\sim \forall y G(y,v) \to H(x)\right)\right]$$

to prenex normal form. (6 marks)

[Total : 25 marks]

Q2. (a) Let p, q, r be atomic statements. Use a truth table or a comparison table to show that

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r.$$ (9 marks)

(b) Simplify the following statement to a logically equivalent statement with no more than TWO(2) logical connectives from the set $\{\sim, \wedge, \vee\}$ by stating the law used in each step of the simplification:

$$(\sim p \wedge q) \vee (\sim p \wedge r) \vee (p \wedge \sim q \wedge r) \vee (q \wedge r).$$ (7 marks)

Q2. (Continued)

(c) Given the following quantified statement:

$$\forall x \forall y \big[((x>0)\wedge(y>0)) \to \big(\sqrt{x+y}=\sqrt{x}+\sqrt{y}\big)\big]. \qquad (*)$$

(i) Translate the quantified statement into an informal English sentence. (2 marks)

(ii) Determine whether the quantified statement is true or false in the domain of real numbers. You need to defend your answer. (2 marks)

(iii) Write down the negation of the quantified statement $(*)$ in prenex normal form. (5 marks)

[Total : 25 marks]

PART B: Answer **ALL** questions.

Q3. (a) Use **truth table** to explain whether the following argument is valid or invalid:

$$\begin{array}{rr} & (p \vee q) \rightarrow (p \wedge q) \\ & \sim (p \vee q) \\ \hline \therefore & \sim (p \wedge q) \end{array}$$

(9 marks)

(b) Infer the argument

$$p \vee q,\ p \rightarrow r,\ \sim s \rightarrow \sim q \vdash r \vee s$$

syntatically by stating the **rules of inference** in each step. (6 marks)

(c) Show that the following argument

$$\begin{array}{rr} & \forall x(F(x) \rightarrow \sim G(x)) \\ & \exists x(H(x) \wedge G(x)) \\ \hline \therefore & \exists x(H(x) \wedge \sim F(x) \end{array}$$

is valid using the rules of logical equivalence and implication. (5 marks)

(d) Let $R(x,y)$ be a predicate with two variables. Infer the argument involving quantified statements

$$\forall x \forall y(R(x,y) \rightarrow \sim R(y,x)) \vdash \forall x(\sim R(x,x))$$

syntatically by stating the **rules of inference** in each step. (5 marks)

[Total : 25 marks]

Q4. (a) Prove by mathematical induction that $17^n - 6^n$ is divisible by 11 for every positive integer n. (8 marks)

(b) Use a proof by contraposition to show that if n is an integer and $n^2 + 5$ is odd, then n is even. (5 marks)

(c) Use the Euclidean algorithm to prove or disprove that $\gcd(198, 54)$ is prime. (4 marks)

Q4. (Continued)

(d) Prove or disprove the following congruence relations.

(i) $-122 \equiv 5 \pmod 7$ (3 marks)

(ii) $3^{2019} \equiv 27 \pmod{40}$ (5 marks)

[Total : 25 marks]

Q5. (a) Let $R = \{(x,y) \in \mathbb{N} \times \mathbb{N} : xy = 1\}$, where $\mathbb{N}$ is the set of positive integers. Determine whether R is reflexive, symmetric, or transitive. Hence, determine whether R is an equivalence relation. Justify your answers. (7 marks)

(b) Let R be the relation on $A = \{1, 2, 5, 6, 7, 11\}$ defined by

$$xRy \text{ if } x \equiv y \pmod 5.$$

Write out the equivalence classes of R and verify that they partition A.

(5 marks)

(c) Let R be a relation defined on the set A whose matrix is

$$M_R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Use M_R to explain why R is not transitive. Then use Warshall's algorithm to find the transitive closure of R. (6 marks)

(d) (i) Define what it means for a relation R on a set A to be a partial order.

(3 marks)

(ii) Let $R = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,c)\}$ a relation on $A = \{a), b, c\}$. Draw the directed graph of R and use it to explain why R is not a partial order. (4 marks)

[Total : 25 marks]

Laws of Logical Equivalence and Implication

Let p, q and r be atomic statements, T be a tautology and F be a contradiction. Suppose the variable x has no free occurrences in ξ and is substitutable for x in ξ. Then

1.	Double negative law:	$\sim(\sim p) \equiv p$.
2.	Idempotent laws:	$p \wedge p \equiv p$; $\quad p \vee p \equiv p$.
3.	Universal bound laws:	$p \vee T \equiv T$; $\quad p \wedge F \equiv F$.
4.	Identity laws:	$p \wedge T \equiv p$; $\quad p \vee F \equiv p$.
5.	Negation laws:	$p \vee \sim p \equiv T$; $\quad p \wedge \sim p \equiv F$.
6.	Commutative laws:	$p \wedge q \equiv q \wedge p$; $\quad p \vee q \equiv q \vee p$.
7.	Absorption laws:	$p \vee (p \wedge q) \equiv p$; $p \wedge (p \vee q) \equiv p$.
8.	Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$; $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
9.	Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$; $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
10.	De Morgan's laws:	$\sim(p \wedge q) \equiv \sim p \vee \sim q$; $\sim(p \vee q) \equiv \sim p \wedge \sim q$.
11.	Implication law:	$p \rightarrow q \equiv \sim p \vee q$
12.	Biconditional law:	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.
13.	Modus Ponens (MP in short):	$p \rightarrow q,\ p \models q$
14.	Modus Tollens (MT in short):	$p \rightarrow q,\ \sim q \models \sim p$
15.	Generalisation:	$p \models p \vee q$; $\ q \models p \vee q$
16.	Specialisation:	$p \wedge q \models p$; $\ p \wedge q \models q$
17.	Conjunction:	$p,\ q \models p \wedge q$
18.	Elimination:	$p \vee q,\ \sim q \models p$; $\ p \vee q,\ \sim p \models q$
19.	Transitivity:	$p \rightarrow q,\ q \rightarrow r \models p \rightarrow r$
20.	Contradiction Rule:	$\sim p \rightarrow F \models p$
21.	Quantified de Morgan laws:	$\sim \forall x \phi \equiv \exists x \sim \phi$; $\sim \exists x \phi \equiv \forall x \sim \phi$;
22.	Quantified conjunctive law:	$\forall x(\phi \wedge \psi) \equiv (\forall x\ \phi) \wedge (\forall x\ \psi)$;
23.	Quantified disjunctive law:	$\exists x(\phi \vee \psi) \equiv (\exists x\ \phi) \vee (\exists x\ \psi)$;
24.	Quantifiers swapping laws:	$\forall x \forall y \phi \equiv \forall y \forall x \phi$; $\quad \exists x \exists y \phi \equiv \exists y \exists x \phi$;
25.	Independent quantifier law:	$\xi \equiv \forall x \xi \equiv \exists x \xi$;
26.	Variable renaming laws:	$\forall x \phi \equiv \forall y \phi[y/x]$; $\quad \exists x \phi \equiv \exists y \phi[y/x]$;
27.	Free variable laws:	$\forall x(\xi \wedge \psi) \equiv \xi \wedge (\forall x \psi)$; $\quad \exists x(\xi \wedge \psi) \equiv \xi \wedge (\exists x \psi)$; $\forall x(\xi \vee \psi) \equiv \xi \vee (\forall x \psi)$; $\quad \exists x(\xi \vee \psi) \equiv \xi \vee (\exists x \psi)$;
28.	Universal instantiation:	$\forall x \phi \Rightarrow \phi[a/x]$;
29.	Universal generalisation:	$\phi[a/x] \Rightarrow \forall x \phi$;
30.	Existential instantiation:	$\exists x \phi \Rightarrow \phi[s/x]$;
31.	Existential generalisation:	$\phi[s/x] \Rightarrow \exists x \phi$.

Rules of Inference

Let ϕ, ψ, ξ be any well-formed formulae. Then

1. $\wedge$-introduction: $\phi, \psi \vdash \phi \wedge \psi$
2. $\wedge$-elimination: $\phi \wedge \psi \vdash \phi$ or $\phi \wedge \psi \vdash \psi$
3. $\rightarrow$-introduction: $\boxed{\phi, \cdots, \psi} \vdash (\phi \rightarrow \psi)$
4. $\rightarrow$-elimination: $\phi \rightarrow \psi, \phi \vdash \psi$
5. $\vee$-introduction: $\phi \vdash \phi \vee \psi$ or $\psi \vdash \phi \vee \psi$
6. $\vee$-elimination: $\phi \vee \psi, \boxed{\phi, \cdots, \xi}, \boxed{\psi, \cdots, \xi} \vdash \xi$
7. $\neg$-introduction or $\sim$-introduction: $\boxed{\sim \phi, \cdots, \bot} \vdash \phi$ or $\boxed{\phi, \cdots, \bot} \vdash \sim \phi$
8. $\neg$-elimination or $\sim$-elimination: $\phi, \sim \phi \vdash \bot$
9. $\forall$-introduction: $\phi(a) \vdash \forall x \phi(x)$
10. $\forall$-elimination: $\forall x \phi(x) \vdash \phi(t)$
11. $\exists$-introduction: $\phi(t) \vdash \exists x \phi(x)$
12. $\exists$-elimination: $\exists x \phi(x), \boxed{\phi(s) \cdots \xi} \vdash \xi$

The term t is free with respect to x in ϕ and $[t/x]$ means "t replaces x".